Math 565: Functional Analysis
Lecture 13

Baire category notions and theorems.

Below let X be a topological space. Recall that a set $D \subseteq X$ is called *dense* in X if every nonempty set $U \subseteq X$ has a member of D in it, i.e. $D \cap U \neq \emptyset$; equivalently, $\overline{D} = X$. In particular, we say that D is dense in some subset $Y \subseteq X$ if $D \cap Y$ is dense in (the relative top. of) Y. If Y is open, then D dense in Y simply means that D meets every nonempty open subset of Y (open in X).

Def. Call a set $D \subseteq X$ *nowhere dense* if for every nonempty open $U \subseteq X$, D is not dense in U. Equivalently, for each nonempty open $U \subseteq X$ there is a further nonempty open $V \subseteq U$ s.t. $D \cap V = \emptyset$.

Prop.
(a) A closed set $C \subseteq X$ is nowhere dense $\iff$ $\mathrm{int}(C) = \emptyset$, i.e. C doesn't contain a $\emptyset \neq$ open set.
(b) A set $D \subseteq X$ is nowhere dense $\iff$ $\overline{D}$ is nowhere dense $\iff$ $\mathrm{int}(\overline{D}) = \emptyset$.

Proof. (a) This follows from the fact that if a closed set is dense in an open set U, then it contains U.
(b) The second equivalence is just part (a) so we only prove the first. $\Leftarrow$ is trivial, and $\Rightarrow$ follows from the fact that if a set D is dense in some open set then $\overline{D}$ contains that open set. $\square$

Examples.
(a) In $\mathbb{R}$ or any other perfect (= no isolated points) metric space, singletons are nowhere dense because they are closed and have empty interior.
(b) In $\mathbb{R}$, $\mathbb{Z}$ is nowhere dense because it is closed and $\mathrm{int}(\mathbb{Z}) = \emptyset$.

(c) In $\mathbb{R}$, $D := \{\frac{1}{n} : n \in \mathbb{N}^+\}$ is nowhere dense because $\overline{D} = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}^+\}$ has empty interior.

(d) In $\mathbb{R}$, every Cantor set (i.e. homeomorphic copy of $2^{\mathbb{N}}$) is nowhere dense because it's closed (actually compact) and has empty interior.

(e) In $\mathbb{R}^n$ every coset of a of k-dimentional subspace, for $k < n$, is nowhere dense because it's closed and has empty interior. For example, lines in $\mathbb{R}^2$ or planes in $\mathbb{R}^3$.

(f) Let $k < \ell \leq \mathbb{N}$, then $k^{\mathbb{N}}$ is a nowhere dense subset of $\ell^{\mathbb{N}}$ because it is closed (points outside of $k^{\mathbb{N}}$ have a finite certificate) and does not contain any cylinder $[w]$ because, $wk^{\infty} \in [w]$ but $wk^{\infty} \notin k^{\mathbb{N}}$.

Upgrade for nowhere dense sets. (a) Every nowhere dense $B \subseteq X$ is contained in a closed nowhere dense set, namely $\overline{B}$.

(a^c) Every co-nowhere-dense set contains an open dense set.

Proof. (a^c) If D is co-nowhere dense then $X \setminus D$ is nowhere dense, hence $\overline{(X \setminus D)}$ is still nowhere dense so $U := X \setminus \overline{(X \setminus D)}$ is dense open since $\overline{(X \setminus D)}$ doesn't contain any nonempty open set. □

Prop. Nowhere dense subsets of X form an ideal, i.e. they are closed downwards under $\subseteq$ and they are closed under finite unions.

Proof. For finite unions, let $A, B \subseteq X$ be nowhere dense and let $U \subseteq X$ be nonempty open. Because A is not dense in U, $\exists$ $\emptyset \neq$ open $V \subseteq U$ with $A \cap V = \emptyset$. Because B is not dense in V, $\exists$ $\emptyset \neq$ open $W \subseteq V$ s.t. $B \cap W = \emptyset$. Hence $(A \cup B) \cap W = \emptyset$. □

We wish to have a notion of smallness that's closed under ctbl unions, i.e. is a σ-ideal. So we define:

Def. A set $B \subseteq X$ is called meagre if B is a ctbl union of nowhere dense sets.

Obs. By definition, meagre sets form a σ-ideal.

Examples. (a) $\mathbb{Q}$ is meagre in $\mathbb{R}$ because $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ is a ctbl union of singletons and each singleton is nowhere dense because it is closed and has empty interior. But $\mathbb{Q}$ is not nowhere dense because it is, in fact, dense in $\mathbb{R}$.

(b) Let $B := \bigcup_{k \geq 2} k^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}}$ is meagre (it's a ctbl union of closed nowhere dense sets), but B itself is dense in $\mathbb{N}^{\mathbb{N}}$ because for any cylinder $[w]$, where $w \in \mathbb{N}^{\ell}$, w has only ℓ many entries so $k := \max_{i < \ell} w(i)$, hence $w 0^{\infty} \in (k+1)^{\mathbb{N}} \cap [w]$.

Upgrade for meagre sets. (a) Every meagre set is contained in an F_σ meagre set, in fact in a ctbl union of closed sets with empty interior.

(a') Every comeagre set contains a G_δ comeagre set, in fact a ctbl intersection of dense open sets.

Analogy with null sets:

meagre $\sim$ null
comeagre $\sim$ conull
nonmeagre $\sim$ nonnull

Remark. This is only an analogy, there is no implication between them. In fact, in $\mathbb{R}$, it is easy to construct a comeagre set (dense G_δ) $D \subseteq \mathbb{R}$ which is Lebesgue null, so D^c is meagre and Lebesgue conull. Thus, measure and category are supported on disjoint sets / are orthogonal. In fact this is true for all atomless Borel measures on separable topological spaces which are outer regular (HW).

As with measure, null only has meaning the measure is nonzero. Similarly, we would like work only in topological spaces in which the whole space is not meagre (so there are nonmeagre). Moreover, as with Lebesgue measure, we would prefer if nonempty open sets were nonmeagre.

Def. A topological space X is called **Baire** if every nonempty open set is nonmeagre.

Caution. Thus, a Baire space is different from the Baire space $\mathbb{N}^{\mathbb{N}}$, although, as we will see, the Baire space is a Baire space.

Prop. For a topological space X, TFAE:

(1) X is Baire, i.e. every nonempty open set is nonmeagre.

(2) Comeagre sets are dense in X.

(3) Ctbl intersections of open dense sets are dense in X.

Proof. (1) $\Rightarrow$ (2). If $D \subseteq X$ is comeagre then D^c is meagre, hence doesn't contain any nonempty open set, so D intersects every nonempty open set, thus is dense.

(2) $\Rightarrow$ (3). Ctbl intersections of open dense sets are comeagre because their complement is a ctbl union of closed sets with empty interior.

(3) $\Rightarrow$ (2). This the upgrade property (a') for meagre sets.

(2) $\Rightarrow$ (1). We show neg(1) $\Rightarrow$ neg(2). If a $\emptyset \neq$ open $U \subseteq X$ is meagre then U^c is comeagre and not dense because it doesn't intersect U. $\square$

Baire Category Theorem. Complete metric spaces are Baire. Also, LcH are Baire.

Proof. We will prove for a complete metric space X, leaving LcH spaces as HW.
Let (U_n) be a sequence of open dense sets. To show $\bigcap_n U_n$ is dense in X, fix a $\emptyset \neq$ open $V_0 \subseteq X$. We will show that $\bigcap_n U_n \cap V_0 \neq \emptyset$ by playing a game:

$$\begin{array}{lccccccc}
\mathrm{I} & V_0 \cap U_0 & & V_1 \cap U_1 & & V_2 \cap U_2 & & \dots \\
 & \rotatebox{90}{$\subseteq$} & \supseteq & \rotatebox{90}{$\subseteq$} & \supseteq & \rotatebox{90}{$\subseteq$} & \supseteq & \\
\mathrm{II} & & V_1 & & V_2 & & V_3 & \dots
\end{array}$$

where at each step Player II plays a nonempty open subset of the open set played by Player I. Then sets $V_n \cap U_n$ are nonempty because V_n is nonempty and U_n is dense. We, i.e. Player II, need to choose V_n so $\bigcap_n V_n \neq \emptyset$. Thus, we let V_n be a ball of diameter $\leq 1/n$ such that $\overline{V_n} \subseteq V_{n-1} \cap U_{n-1}$. Then $\bigcap_{n \in \mathbb{N}} V_n = \bigcap_{n \in \mathbb{N}} \overline{V_n} \neq \emptyset$ by the completeness of the metric space. Thus, $V_0 \cap \bigcap_{n \in \mathbb{N}} U_n \supseteq \bigcap_n \overline{V_n} \neq \emptyset$. $\square$